

# Congruent conditions on the number of terms, on the ratio number of terms to first terms and on the difference of first terms for sums of consecutive squared integers equal to squared integers

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## Abstract

Sums of  $M$  consecutive squared integers  $(a + i)^2$  equaling squared integers (for  $a \geq 1$ ,  $0 \leq i \leq M - 1$ ) yield certain linear groupings of pairs  $(a_1, a_2)$  of  $a$  values for successive same values of  $M$  when these are linked by  $a_1 + a_2 = \mu M + 1$  with  $\mu = (\eta/\delta) \in \mathbb{Q}^+$ . In this paper, congruent conditions on  $M, \eta, \delta$ , and on the difference  $(a_2 - a_1)$  are demonstrated for these linear groupings to hold. It is found that  $\eta \equiv 1 \pmod{2}$  and  $\delta \equiv 0, 1$  or  $5 \pmod{6}$ , and if  $\delta \equiv 0 \pmod{6}$ ,  $M \equiv 0 \pmod{12}$ , while if  $\delta \equiv 1$  or  $5 \pmod{6}$ ,  $M \equiv 2$  or  $11 \pmod{12}$  with  $a_1$  and  $a_2$  being of different or same parities.

**Keywords:** Sums of consecutive squared integers equal to square integers  
; Congruence

MSC2010 : 11E25 ; 11A07

## 1 Introduction

Finding all values of  $a \geq 1$  for which the sum of  $M > 1$  consecutive integer squares starting from  $a^2$  equals an integer square  $s^2$  was addressed by several authors since Lucas proposed the initial problem for  $a = 1$  in 1873 (see e.g. [4, 5, 6, 1, 7, 3, 2]). More recently, the present author showed [8, 9] that there are no integer solutions if  $M \equiv 3, 5, 6, 7, 8$  or  $10 \pmod{12}$ ; that there are integer solutions if  $M$  is not a square integer and is congruent to  $0, 9, 24$  or  $33 \pmod{72}$ ,

or to 1, 2 or 16 (mod 24), or to 11 (mod 12); and if  $M$  is a square integer and congruent to 1 (mod 24).

Interestingly, for certain values of  $a$  and  $M$ , one finds linear groupings of pairs  $(a_1, a_2)$  of  $a$  values for successive same values of  $M$  when these values are linked by the relation  $a_1 + a_2 = \mu M + 1$  with  $\mu = (\eta/\delta) \in \mathbb{Q}^+$  [10].

In this paper, congruent conditions on  $M, \eta, \delta$ , and on the difference  $(a_2 - a_1)$  are demonstrated for these groupings to hold.

## 2 Pairs of $a$ values

For  $1 \leq j \leq 2, i, \eta, \delta, M_{\mu,k} > 1, a_{j,\mu,k} \in \mathbb{Z}^*, k, s_{j,\mu,k} \in \mathbb{Z}$ , let  $\mu = (\eta/\delta) \in \mathbb{Q}^+$  forming an irreducible fraction. Two values  $a_{1,\mu,k}, a_{2,\mu,k}$  form a pair of  $a_{j,\mu,k}$  values for a same value of  $M_{\mu,k}$  if

$$a_{1,\mu,k} + a_{2,\mu,k} = \mu M_{\mu,k} + 1 \quad (1)$$

$$a_{2,\mu,k} - a_{1,\mu,k} = f_{\mu,k} \quad (2)$$

It was demonstrated [10] that if

$$M_{\mu,k} = \frac{\delta^2 (3f_{\mu,k}^2 - 1)}{3(\eta + \delta)^2 + \delta^2} \quad (3)$$

then the sums of  $M_{\mu,k}$  consecutive squared integers  $(a_{j,\mu,k} + i)^2$  always equal squared integers  $s_{j,\mu,k}^2$  for pairs of  $a_{j,\mu,k}$  values

$$\sum_{i=0}^{M-1} (a_{j,\mu,k} + i)^2 = M_{\mu,k} \left[ \left( a + \frac{M_{\mu,k} - 1}{2} \right)^2 + \frac{M_{\mu,k}^2 - 1}{12} \right] \quad (4)$$

$\forall k \in \mathbb{Z}$  and where  $j = 1$  or 2.

## 3 Congruent values of $M_{\mu,k}, f_{\mu,k}$ and $\mu = (\eta/\delta)$

These results hold only for certain allowed values of  $M_{\mu,k}$ , of  $f_{\mu,k}$  and of  $\mu = (\eta/\delta) \in \mathbb{Q}^+$ , that can be determined as follows. Relation (3) reads also

$$(\delta f_{\mu,k})^2 - M_{\mu,k} (\eta + \delta)^2 = \delta^2 \left( \frac{M_{\mu,k} + 1}{3} \right) \quad (5)$$

As the sum of  $M$  consecutive integer squares can only be equal to a squared integer if  $M \equiv 0 \pmod{12}$  (more precisely  $M \equiv 0$  or  $24 \pmod{72}$ ) or if  $M \equiv 1, 2$  or  $4 \pmod{12}$  (more precisely  $M \equiv 1, 2$  or  $16 \pmod{24}$ ), or if  $M \equiv 9 \pmod{12}$  (more precisely  $M \equiv 9$  or  $33 \pmod{72}$ ), or if  $M \equiv 11 \pmod{12}$  [8, 9], the following theorem constrains the congruent values of  $\eta, \delta, M_{\mu,k}$  and  $f_{\mu,k}$ .

**Theorem 1.** For  $\eta, \delta, M_{\mu,k} > 1, f_{\mu,k} \in \mathbb{Z}^+, k \in \mathbb{Z}$ , for (5) to hold:  $\eta \equiv 1 \pmod{2}$  and  $\delta \equiv 0, 1$  or  $5 \pmod{6}$ , and

Table 1: Congruent values of  $\eta, \delta, M_{\mu,k}$  and  $f_{\mu,k}$  for  $\delta \equiv 0 \pmod{6}$  and  $\delta \equiv 1$  or  $5 \pmod{6}$

$\delta \equiv$	$\eta \equiv$	$f_{\mu,k} \equiv$	$M_{\mu,k} \equiv$
$0 \pmod{36}$	$1 \text{ or } 5 \pmod{6}$	$\forall$	$0 \pmod{144}$
$12 \text{ or } 24 \pmod{36}$			$96 \pmod{144}$
$6 \text{ or } 30 \pmod{36}$	$1 \text{ or } 5 \pmod{6}$	$1 \pmod{2}$	$24 \pmod{144}$
$18 \pmod{36}$			$72 \pmod{144}$
$1 \pmod{6}$	$1 \text{ or } 3 \pmod{6}$	$1 \text{ or } 5 \pmod{6}$	$50 \pmod{72}$
	$5 \pmod{6}$		$2 \pmod{72}$
	$1 \text{ or } 3 \pmod{6}$	$3 \pmod{6}$	$2 \pmod{72}$
	$5 \pmod{6}$		$26 \pmod{72}$
$5 \pmod{6}$	$1 \pmod{6}$	$1 \text{ or } 5 \pmod{6}$	$2 \pmod{72}$
	$3 \text{ or } 5 \pmod{6}$		$50 \pmod{72}$
	$1 \pmod{6}$	$3 \pmod{6}$	$26 \pmod{72}$
	$3 \text{ or } 5 \pmod{6}$		$2 \pmod{72}$
$1 \pmod{6}$	$1 \text{ or } 3 \pmod{6}$	$0 \pmod{6}$	$11 \pmod{36}$
	$5 \pmod{6}$		$35 \pmod{36}$
	$1 \text{ or } 3 \pmod{6}$	$2 \text{ or } 4 \pmod{6}$	$23 \pmod{36}$
	$5 \pmod{6}$		$11 \pmod{36}$
$5 \pmod{6}$	$1 \pmod{6}$	$0 \pmod{6}$	$35 \pmod{36}$
	$3 \text{ or } 5 \pmod{6}$		$11 \pmod{36}$
	$1 \pmod{6}$	$2 \text{ or } 4 \pmod{6}$	$11 \pmod{36}$
	$3 \text{ or } 5 \pmod{6}$		$23 \pmod{36}$

i.e. for  $\delta \equiv 0 \pmod{36}$  and  $\eta \equiv 1 \text{ or } 5 \pmod{6}$ ,  $\forall f_{\mu,k}$  and  $M_{\mu,k} \equiv 0 \pmod{144}$ ;  
for  $\delta \equiv 1 \pmod{6}$  and  $\eta \equiv 1 \text{ or } 3 \pmod{6}$ ,  $f_{\mu,k} \equiv 1 \text{ or } 5 \pmod{6}$  and  
 $M_{\mu,k} \equiv 50 \pmod{72}$

- if  $\delta \equiv 0 \pmod{6}$ ,  $M_{\mu,k} \equiv 0 \pmod{12}$ ,
  - if  $\delta \equiv 1$  or  $5 \pmod{6}$ ,  $M_{\mu,k} \equiv 2$  or  $11 \pmod{12}$  for  $f_{\mu,k} \equiv 1$  or  $0 \pmod{2}$ ;
- more precisely,  $\eta, \delta, M_{\mu,k}$  and  $f_{\mu,k}$  are congruent to the values of Table 1.

*Proof.* For  $\eta, \delta, d, M_{\mu,k} > 1, f_{\mu,k}, m, n \in \mathbb{Z}^+, \lambda \in \mathbb{Z}^*, k \in \mathbb{Z}$ , recalling that  $n^2 \equiv 0, 1, 4$  or  $9 \pmod{12}, \forall n \in \mathbb{Z}^+$ , and that  $M_{\mu,k} \equiv 0, 1, 2, 4, 9$  or  $11 \pmod{12}$  [8],

(i) Let  $\delta \equiv 0 \pmod{6} \Rightarrow \delta^2 \equiv 0 \pmod{12}$ , then only  $(\eta + \delta)^2 \equiv 1 \pmod{12}$  holds for, if  $(\eta + \delta)^2 \equiv 0, 4$  or  $9 \pmod{12}, \Rightarrow (\eta + \delta) \pmod{6} \equiv \eta \pmod{6} \equiv 0, (2, 4)$  or  $3 \pmod{6}$  as  $\delta \equiv 0 \pmod{6}$  and  $(\eta/\delta)$  would not be an irreducible fraction; therefore  $(\eta + \delta)^2 \equiv 1 \pmod{12}$  and  $\eta \equiv 1$  or  $5 \pmod{6}$ .

Then (5) yields the congruence relation  $M_{\mu,k} \equiv 0 \pmod{12}$ , and more precisely  $M_{\mu,k} \equiv 0$  or  $24 \pmod{72}$  [8], i.e.  $\exists m \in \mathbb{Z}^+$  such that  $M_{\mu,k} = 24(3m + \lambda)$ , with  $\lambda = 0$  or  $1$  if  $M_{\mu,k} \equiv 0$  or  $24 \pmod{72}$ .

As  $\delta \equiv 0 \pmod{6}, \exists d \in \mathbb{Z}^+$  such that  $\delta = 6d$  and  $\delta^2 = 36d^2$ . Then (5) reads after dividing by 12,

$$3d^2 f_{\mu,k}^2 - 2(3m + \lambda)(\eta + 6d)^2 = d^2(24(3m + \lambda) + 1) \quad (6)$$

that yields the congruence relation

$$3d^2 f_{\mu,k}^2 - 2(3m + \lambda) - d^2 \equiv 0 \pmod{12} \quad (7)$$

(i.1) If  $d^2 \equiv 0 \pmod{12}$ , i.e.  $d \equiv 0 \pmod{6}$ , then (7) reduces to  $10(3m + \lambda) \equiv 0 \pmod{12} \Rightarrow (3m + \lambda) \equiv 0 \pmod{6}$  which cannot hold if  $\lambda = 1$  and which yields  $m \equiv 0 \pmod{2}$  if  $\lambda = 0$ , i.e.  $M_{\mu,k} \equiv 0 \pmod{144}$ , and  $\forall f_{\mu,k}$  with  $\delta \equiv 0 \pmod{36}$ .

(i.2) If  $d^2 \equiv 1 \pmod{12}$ , i.e.  $d \equiv 1$  or  $5 \pmod{6}$ , then (7) reads

$$3f_{\mu,k}^2 - 2(3m + \lambda) \equiv 1 \pmod{12} \quad (8)$$

(i.2.1) If  $f_{\mu,k}^2 \equiv 0$  or  $4 \pmod{12}$ , then (8) reduces to  $10(3m + \lambda) \equiv 1 \pmod{12}$  which cannot hold whether  $\lambda = 0$  or  $1$ ;

(i.2.2) if  $f_{\mu,k}^2 \equiv 1$  or  $9 \pmod{12}$ , then (8) reduces to  $2(3m + \lambda) \equiv 2 \pmod{12} \Rightarrow (3m + \lambda) \equiv 1 \pmod{6}$  which cannot hold if  $\lambda = 0$  and which yields  $m \equiv 0 \pmod{2}$  if  $\lambda = 1$ , i.e.  $M_{\mu,k} \equiv 24 \pmod{144}$ , and  $f_{\mu,k} \equiv 1 \pmod{2}$  with  $\delta \equiv 6$  or  $30 \pmod{36}$ .

(i.3) If  $d^2 \equiv 4 \pmod{12}$ , i.e.  $d \equiv 2$  or  $4 \pmod{6}$ , then (7) reads  $10(3m + \lambda) \equiv 4 \pmod{12} \Rightarrow (3m + \lambda) \equiv 4 \pmod{6}$  which cannot hold if  $\lambda = 0$  and which yields  $m \equiv 1 \pmod{2}$  if  $\lambda = 1$ , i.e.  $M_{\mu,k} \equiv 96 \pmod{144}$ , and  $\forall f_{\mu,k}$  with  $\delta \equiv 12$  or  $24 \pmod{36}$ .

(i.4) If  $d^2 \equiv 9 \pmod{12}$ , i.e.  $d \equiv 3 \pmod{6}$ , then (7) reads

$$3f_{\mu,k}^2 - 2(3m + \lambda) \equiv 9 \pmod{12} \quad (9)$$

(i.4.1) If  $f_{\mu,k}^2 \equiv 0$  or  $4 \pmod{12}$ , then (9) reduces to  $10(3m + \lambda) \equiv 9 \pmod{12}$  which cannot hold whether  $\lambda = 0$  or  $1$ ;

(i.4.2) if  $f_{\mu,k}^2 \equiv 1$  or  $9 \pmod{12}$ , then (9) reduces to  $2(3m + \lambda) \equiv 6 \pmod{12} \Rightarrow (3m + \lambda) \equiv 3 \pmod{6}$  which cannot hold if  $\lambda = 1$  and which yield  $m \equiv 1 \pmod{2}$  if  $\lambda = 0$ , i.e.  $M_{\mu,k} \equiv 72 \pmod{144}$ , and  $f_{\mu,k} \equiv 1 \pmod{2}$  with  $\delta \equiv 18 \pmod{36}$ .  
(ii) Let now  $\delta \equiv 3 \pmod{6} \Rightarrow \delta^2 \equiv 9 \pmod{12}$ , then only  $(\eta + \delta)^2 \equiv 1$  or  $4 \pmod{12}$  hold for, if  $(\eta + \delta)^2 \equiv 0$  or  $9 \pmod{12}$ ,  $\Rightarrow (\eta + \delta) \equiv 0$  or  $3 \pmod{6} \Rightarrow \eta \equiv 3$  or  $0 \pmod{6}$  as  $\delta \equiv 3 \pmod{6}$  and  $(\eta/\delta)$  would not be an irreducible fraction; therefore  $(\eta + \delta)^2 \equiv 1$  or  $4 \pmod{12}$ .  
Relation (5) yields then the congruence relation

$$9f_{\mu,k}^2 - M_{\mu,k} \left( (\eta + \delta)^2 + 3 \right) \equiv 3 \pmod{12} \quad (10)$$

(ii.1) If  $(\eta + \delta)^2 \equiv 1 \pmod{12}$ , (10) reduces to  $9f_{\mu,k}^2 - 4M_{\mu,k} \equiv 3 \pmod{12}$  which cannot hold whether  $f_{\mu,k}^2 \equiv 0, 1, 4$  or  $9 \pmod{12}$ .  
(ii.2) If  $(\eta + \delta)^2 \equiv 4 \pmod{12}$ , (10) yields  $9f_{\mu,k}^2 - 7M_{\mu,k} \equiv 3 \pmod{12}$ ;  
(ii.2.1) if  $f_{\mu,k}^2 \equiv 0$  or  $4 \pmod{12}$ , it reduces to  $M_{\mu,k} \equiv 3 \pmod{12}$ ;  
(ii.2.2) if  $f_{\mu,k}^2 \equiv 1$  or  $9 \pmod{12}$ , it reduces to  $M_{\mu,k} \equiv 6 \pmod{12}$ ;  
both cases have to be rejected as  $M_{\mu,k}$  cannot be congruent to 3 or 6  $\pmod{12}$  [8].  
(iii) If  $\delta \equiv 1$  or  $2 \pmod{3}$ , then for (5) to hold in integer values of  $\eta, \delta, M_{\mu,k}$  and  $f_{\mu,k}$ ,  $M_{\mu,k} \equiv 2 \pmod{3}$  for the right hand term of (5) to be integer. As  $M_{\mu,k}$  can only be congruent to 0, 1, 2, 4, 9 or 11  $\pmod{12}$ ,  $M_{\mu,k} \equiv 2 \pmod{3}$  can only be  $M_{\mu,k} \equiv 2 \pmod{12}$  (and more precisely  $M_{\mu,k} \equiv 2 \pmod{24}$ ) or  $M_{\mu,k} \equiv 11 \pmod{12}$  [8]. It yields then that :  
if  $M_{\mu,k} \equiv 2 \pmod{24}$ ,  $((M_{\mu,k} + 1)/3) \equiv 1, 9$  or  $5 \pmod{12}$  for  $M_{\mu,k} \equiv 2, 26$  or  $50 \pmod{72}$ , and  
if  $M_{\mu,k} \equiv 11 \pmod{12}$ ,  $((M_{\mu,k} + 1)/3) \equiv 4, 8$  or  $0 \pmod{12}$  for  $M_{\mu,k} \equiv 11, 23$  or  $35 \pmod{36}$ .  
(iii.1) Let  $\delta^2 \equiv 1 \pmod{12}$ , i.e.  $\delta \equiv 1$  or  $5 \pmod{6}$ , then (5) yields

$$M_{\mu,k} (\eta + \delta)^2 + \left( \frac{M_{\mu,k} + 1}{3} \right) - f_{\mu,k}^2 \equiv 0 \pmod{12} \quad (11)$$

and one obtains as follows one of the two solutions :

Solution 1:  $(\eta + \delta)^2 \equiv 0 \pmod{12} \Rightarrow (\eta + \delta) \equiv 0 \pmod{6}$

$\Rightarrow \eta \equiv 5 \pmod{6}$  or 1 as  $\delta \equiv 1$  or  $5 \pmod{6}$ ,

Solution 2:  $(\eta + \delta)^2 \equiv 4 \pmod{12} \Rightarrow (\eta + \delta) \equiv 2$  or  $4 \pmod{6}$

$\Rightarrow \eta \equiv 1$  or  $3 \pmod{6}$  if  $\delta \equiv 1 \pmod{6}$  and  $\eta \equiv 3$  or  $5 \pmod{6}$  if  $\delta \equiv 5 \pmod{6}$ .

(iii.1.1) Let  $f_{\mu,k}^2 \equiv 0$  or  $4 \pmod{12}$ , i.e.  $f_{\mu,k} \equiv 0 \pmod{2}$ , then :

- if  $M_{\mu,k} \equiv 2 \pmod{12}$  with  $((M_{\mu,k} + 1)/3) \equiv 1, 5$  or  $9 \pmod{12}$ , (11) cannot hold;

- if  $M_{\mu,k} \equiv 11 \pmod{12}$  and

— if  $((M_{\mu,k} + 1)/3) \equiv 0 \pmod{12}$  (for  $M_{\mu,k} \equiv 35 \pmod{36}$ ), (11) yields Solution 1 for  $f_{\mu,k}^2 \equiv 0 \pmod{12}$  and  $(\eta + \delta)^2 \equiv 8 \pmod{12}$ , which cannot hold, for  $f_{\mu,k}^2 \equiv 4 \pmod{12}$ ;

— if  $((M_{\mu,k} + 1)/3) \equiv 4 \pmod{12}$  (for  $M_{\mu,k} \equiv 11 \pmod{36}$ ), (11) yields Solution 2 or 1 for  $f_{\mu,k}^2 \equiv 0$  or  $4 \pmod{12}$ ;

— if  $((M_{\mu,k} + 1)/3) \equiv 8 \pmod{12}$  (for  $M_{\mu,k} \equiv 23 \pmod{36}$ ), (11) yields  $(\eta + \delta)^2 \equiv 8 \pmod{12}$ , which cannot hold, for  $f_{\mu,k}^2 \equiv 0 \pmod{12}$  and Solution 2 for  $f_{\mu,k}^2 \equiv 4 \pmod{12}$ .

(iii.1.2) Let  $f_{\mu,k}^2 \equiv 1$  or  $9 \pmod{12}$ , i.e.  $f_{\mu,k} \equiv 1 \pmod{2}$ , then:

- if  $M_{\mu,k} \equiv 2 \pmod{12}$  and

— if  $((M_{\mu,k} + 1)/3) \equiv 1 \pmod{12}$  (for  $M_{\mu,k} \equiv 2 \pmod{72}$ ), (11) yields Solution 1 or 2 for  $f_{\mu,k}^2 \equiv 1$  or  $9 \pmod{12}$ ;

— if  $((M_{\mu,k} + 1)/3) \equiv 5 \pmod{12}$  (for  $M_{\mu,k} \equiv 50 \pmod{72}$ ), (11) yields Solution 2 for  $f_{\mu,k}^2 \equiv 1 \pmod{12}$  and  $(\eta + \delta)^2 \equiv 2$  or  $8 \pmod{12}$ , which cannot hold, for  $f_{\mu,k}^2 \equiv 9 \pmod{12}$ ;

— if  $((M_{\mu,k} + 1)/3) \equiv 9 \pmod{12}$  (for  $M_{\mu,k} \equiv 26 \pmod{72}$ ), (11) yields  $(\eta + \delta)^2 \equiv 2$  or  $8 \pmod{12}$ , which cannot hold, for  $f_{\mu,k}^2 \equiv 1 \pmod{12}$  and Solution 1 for  $f_{\mu,k}^2 \equiv 9 \pmod{12}$ .

- if  $M_{\mu,k} \equiv 11 \pmod{12}$  with  $((M_{\mu,k} + 1)/3) \equiv 0, 4$  or  $8 \pmod{12}$ , then (11) cannot hold.

(iii.2) Let now  $\delta^2 \equiv 4 \pmod{12}$ , i.e.  $\delta \equiv 2$  or  $4 \pmod{6}$ , then (5) yields

$$M_{\mu,k} (\eta + \delta)^2 + 4 \left( \frac{M_{\mu,k} + 1}{3} \right) - 4f_{\mu,k}^2 \equiv 0 \pmod{12} \quad (12)$$

and one obtains as follows one of the two solutions, both to be rejected as  $(\eta/\delta)$  would not be an irreducible fraction :

Solution 3:  $(\eta + \delta)^2 \equiv 0 \pmod{12} \Rightarrow (\eta + \delta) \equiv 0 \pmod{6}$

$\Rightarrow \eta \equiv 4$  or  $2 \pmod{6}$  as  $\delta \equiv 2$  or  $4 \pmod{6}$ ,

Solution 4:  $(\eta + \delta)^2 \equiv 4 \pmod{12} \Rightarrow (\eta + \delta) \equiv 2$  or  $4 \pmod{6}$

$\Rightarrow \eta \equiv 0$  or  $2 \pmod{6}$  if  $\delta \equiv 2 \pmod{6}$  and  $\eta \equiv 4$  or  $2 \pmod{6}$  if  $\delta \equiv 4 \pmod{6}$ .

(iii.2.1) Let  $f_{\mu,k}^2 \equiv 0$  or  $9 \pmod{12}$ , then:

- if  $M_{\mu,k} \equiv 2 \pmod{12}$  and

— if  $((M_{\mu,k} + 1)/3) \equiv 1 \pmod{12}$ , (12) yields Solution 4;

— if  $((M_{\mu,k} + 1)/3) \equiv 5$  or  $9 \pmod{12}$ , (12) cannot hold;

- if  $M_{\mu,k} \equiv 11 \pmod{12}$  and

— if  $((M_{\mu,k} + 1)/3) \equiv 0 \pmod{12}$ , (12) yields Solution 3

— if  $((M_{\mu,k} + 1)/3) \equiv 4$  or  $8 \pmod{12}$ , (12) cannot hold.

(iii.2.2) Let  $f_{\mu,k}^2 \equiv 1$  or  $4 \pmod{12}$ , then:

- if  $M_{\mu,k} \equiv 2 \pmod{12}$  and

— if  $((M_{\mu,k} + 1)/3) \equiv 1 \pmod{12}$ , (12) yields Solution 3;

— if  $((M_{\mu,k} + 1)/3) \equiv 5$  or  $9 \pmod{12}$ , (12) cannot hold.

- if  $M_{\mu,k} \equiv 11 \pmod{12}$  and

— if  $((M_{\mu,k} + 1)/3) \equiv 0 \pmod{12}$ , (12) cannot hold;

— if  $((M_{\mu,k} + 1)/3) \equiv 4$  or  $8 \pmod{12}$ , (12) yields Solution 3 or 4.

Therefore, the congruence of Table 1 hold.  $\square$

The following congruent relations for  $M_{\mu,k}$  also hold.

**Corollary 2.** For  $\eta, \delta, M_{\mu,k} > 1, f_{\mu,k} \in \mathbb{Z}^+, k \in \mathbb{Z}$ :

- (i) if  $\delta \equiv 1 \text{ or } 5 \pmod{6}$ , then  $M_{\mu,k} \equiv 0 \pmod{\delta^2}$ ;  
 if  $\delta \equiv 0 \pmod{6}$ , then  $M_{\mu,k} \equiv 0 \pmod{(\delta^2/3)}$ ;
- (ii) if  $f_{\mu,k} \equiv 1 \pmod{2}$ , then, if  $\delta \equiv 1 \text{ or } 5 \pmod{6}$ ,  $M_{\mu,k} \equiv 0 \pmod{2\delta^2}$  and,  
 if  $\delta \equiv 0 \pmod{6}$ , then  $M_{\mu,k} \equiv 0 \pmod{(2\delta^2/3)}$ .

*Proof.* For  $\eta, \delta, M_{\mu,k} > 1, f_{\mu,k} \in \mathbb{Z}^+, k \in \mathbb{Z}$ :

- (i) from (3), as  $\gcd(\eta, \delta) = 1$  and as  $M_{\mu,k}$  must be integer, if  $\delta \equiv 1 \text{ or } 5 \pmod{6}$ , then  $\left(3(\eta + \delta)^2 + \delta^2\right)$  must divide  $\left(3f_{\mu,k}^2 - 1\right)$ , therefore  $M_{\mu,k} \equiv 0 \pmod{\delta^2}$ ;  
 if  $\delta \equiv 0 \pmod{6}$ , then  $\left((\eta + \delta)^2 + (\delta^2/3)\right)$  must divide  $\left(3f_{\mu,k}^2 - 1\right)$ , therefore  $M_{\mu,k} \equiv 0 \pmod{(\delta^2/3)}$ ;
- (ii) immediate from (i) and Theorem 1. □

## 4 Conclusions

For pairs  $(a_{1,\mu,k}, a_{2,\mu,k})$  of  $a$  values, the sums of  $M_{\mu,k}$  consecutive squared integers starting with  $a_{1,\mu,k}$  or  $a_{2,\mu,k}$  are always equal to squared integers  $s_{1,\mu,k}^2$  or  $s_{2,\mu,k}^2 \forall k \in \mathbb{Z}$  if  $M_{\mu,k} = \delta^2 \left(3(a_{2,\mu,k} - a_{1,\mu,k})^2 - 1\right) / \left(3(\eta + \delta)^2 + \delta^2\right)$  with  $\mu = (\eta/\delta) \in \mathbb{Q}^+$ . It was proved that  $\eta \equiv 1 \pmod{2}$  and  $\delta \equiv 0, 1 \text{ or } 5 \pmod{6}$  and in addition :

- if  $\delta \equiv 0 \pmod{6}$ , then  $M_{\mu,k} \equiv 0 \pmod{12}$  (more precisely,  $M_{\mu,k} \equiv 0, 24, 72 \text{ or } 96 \pmod{144}$ ), and if  $\delta \equiv 1 \text{ or } 5 \pmod{6}$ , depending on  $a_{1,\mu,k}$  and  $a_{2,\mu,k}$  having different or same parities, then  $M_{\mu,k} \equiv 2 \text{ or } 11 \pmod{12}$  (more precisely,  $M_{\mu,k} \equiv 2, 26 \text{ or } 50 \pmod{72}$  or  $M_{\mu,k} \equiv 11, 23 \text{ or } 35 \pmod{36}$ );
- if  $\delta \equiv 1 \text{ or } 5 \pmod{6}$ , then  $M_{\mu,k} \equiv 0 \pmod{\delta^2}$  and, if in addition  $a_{1,\mu,k}$  and  $a_{2,\mu,k}$  have different parities,  $M_{\mu,k} \equiv 0 \pmod{2\delta^2}$  ; if  $\delta \equiv 0 \pmod{6}$ , then  $M_{\mu,k} \equiv 0 \pmod{(\delta^2/3)}$  and, if in addition  $a_{1,\mu,k}$  and  $a_{2,\mu,k}$  have different parities,  $M_{\mu,k} \equiv 0 \pmod{(2\delta^2/3)}$ .

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